CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Electrical Engineering
Division of Systems and Control

EXAMINATION IN LINEAR CONTROL SYSTEM DESIGN

(Course SSY285)

Monday August 21, 2025

Time and place:	08:30 - 12:30 (Johanneberg)
Teacher:	Torsten Wik (031 - 772 5146)

The total points achievable are 30. Unfinished solutions normally results in 0 points. Incorrect solutions with significant errors, unrealistic results that are not commented, or solutions that are too difficult to follow also result in 0 points.

Numerically incorrect calculations that do not cause unrealistic results will normally lead to reduction of 1 point. Incorrect answers that are consequences of a previous error and do not simplify the problem will not lead to any further reduction.

The scales for grading are

Grade 3:	at least 12 points
Grade 4:	at least 18 points
Grade 5:	at least 24 points

The following aids are allowed:

1. Course text book *Control Theory* (or Swedish version *Reglerteori*) by T. Glad and L. Ljung and **one more** control textbook.

   Paper copies are accepted instead of books.

2. 1 piece of A4 paper, with *hand written* notes on both sides. Copied sheets are not allowed!

3. Mathematical and physical handbooks of tables, such as Physics handbook and Beta Mathematics Handbook.

4. Memory depleted, non-programmable pocket calculator.

Solutions to problems and exercises are not allowed in the notes!

Mobile telephones, laptops or tablets/iPads are not allowed!

Good Luck!

1. Decide if the following statements for linear time-invariant systems are *always* true or not (short motivation is required):

   (a) If a controllable continuous time system is discretized, then the corresponding discrete time system is controllable.

   (b) If a continuous time system is uncontrollable, then the corresponding discrete time system is also uncontrollable.

2 p.

2. Assume that we have a process described by

$$\begin{aligned} x(k+1) &= ax(k) + bu(k) + v_1(k) \\ y(k) &= cx(k) + v_2(k) \end{aligned}$$

   where v_1 and v_2 are independent white zero mean Gaussian noise with variances σ_1^2 and σ_2^2. The system parameters are $a = 1/\sqrt{2}$, $b = 1$ and $c = 1/\sqrt{2}$.

   Note: This problem contains many questions. However, many of them can be solved independently of each other, so please read carefully through all questions!

   (a) Determine the stationary observer giving the smallest achievable variance of the estimation error $x(k) - \hat{x}(k)$, where the estimate is based on the current measurement $y(k)$, i.e. $\hat{x}(k) = \hat{x}(k|k)$.

4 p.

   (b) If $\sigma_1^2 = 1$ and $\sigma_2^2 = 1$, how large is the variance of the estimation error?

1 p.

   (c) What happens to the pole of the filter when the measurement noise increases to become very large compared to σ_1^2? Give a "physical" interpretation/explanation.

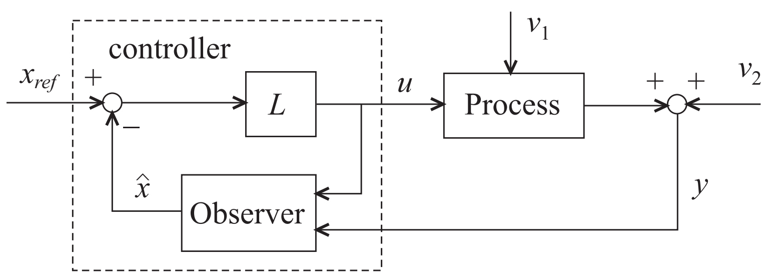
3 p.

   (d) The estimate will be used for state feedback control minimizing

$$\text{Var}\{x - x_{ref}\} + 2\text{Var}\{u\} \tag{1}$$

   The controller is illustrated in the figure below. Determine the feedback gain L!

3 p.



(e) Suppose the resulting controller resulted in too large control signals. Suggest a modification of the criterion (1)!

1 p.

(f) It turns out that, in addition to the zero mean process disturbance v, there are load disturbances that cause stationary errors in the control. To eliminate these stationary errors an additional integral state

$$x_I(k) = \frac{1}{q-1}(x_{ref} - \frac{1}{c}y(k))$$

is introduced. The integral action is tuned by modifying the weight α on the integral state in the new criterion to be minimized

$$V = \text{Var}\{x - x_{ref}\} + \alpha \text{Var}\{x_I\} + \beta \text{Var}\{u\}$$

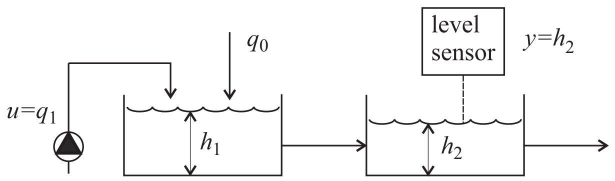
Give all equations and matrices needed for the determination of the new control law (you need not solve them!)

2 p.

(g) Show how the block diagram of the controller is modified with the above integral action.

1 p.

3. Consider the system below with two tanks in series, where the purpose is to keep the level h_2 in the second tank at given setpoints. To the first tank we have a flow q_0 that is fairly constant, but needs to be complemented with a flow q_1 that we can control using a pump.



In the operating point the transfer function from the control signal u to the level h_1 in the first tank is given by

$$H_1(s) = \frac{2}{1+s}U(s)$$

and the transfer function from the level in the first tank to the level in the second tank is given by

$$H_2(s) = \frac{1}{1+2s}H_1(s)$$

Currently, only the level in tank 2 is measured ($y = h_2$), but it is still important to ensure that the level h_1 in the first tank is neither too low nor too high. We shall now investigate wether a level sensor needs to be installed in the first tank or if we can estimate the level based on the measurement in the second tank.

(Note that all problems can be solved individually!)

(a) Show that the system from u till $y = h_2$ can be put on the state-space form

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0.5 & -0.5 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned}$$

2 p.

(b) Is the system observable?

1 p.

(c) Derive an observer on the form

$$\frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

such that the observer gets a double pole in -1.

2 p.

4. A continuous time scalar system can be written

$$\begin{aligned} \frac{d}{dt}x(t) &= u(t) + w(t) \\ y(t) &= x(t) + n(t) \end{aligned}$$

The disturbances w and n are uncorrelated with zero mean and have the spectra

$$\Phi_w(\omega) = \frac{4}{\omega^2 + 1} \qquad \Phi_n(\omega) = \frac{\omega^2 + 1}{\omega^2 + 4}$$

The system is to be optimally controlled and therefore we want to write the system on the form

$$\begin{aligned} \frac{d}{dt}x_e(t) &= Ax_e(t) + Bu(t) + Nv_1(t) \\ y(t) &= Cx_e(t) + Du(t) + v_2(t) \end{aligned}$$

where $v = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$ is white noise with zero mean and intensity

$$R = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

Determine such a model and state what the intensity matrix R is.

4 p.

5. Assume we have the following perfect discrete-time n:th order LTI model of a process and no disturbances

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

(a) Calculate the inputs $u(0)$ and $u(1)$ that brings the state to the origin in 2 samples from any initial x(0).

2 p.

(b) *Derive* the requirements on general matrices A (n by n) and B (n by m) of the model to bring the state from any arbitrary state to any other arbitrary state in n samples. What is this kind of controller called?

2 p.