Solutions to SSY285 2025-08-21

1. (a) No. When you go from continuous time to discrete time you cannot achieve the same spectrum of control actuation and thus you may loose controllability

   (b) For the same reason, if you cannot control the system with continuous time control actuation you cannot do it when you limit the actuation to be discrete time (usually zero order hold).

2.

a) The optimal observer is the stationary Kalman filter given by (5.99-5.102) with $A=a$, $B=b$, $C=c$, $R_1=\sigma_1^2$, $R_2=\sigma_2^2$ and K, $\tilde{K}$, $P(k+1|k)=P$ and $P(k|k)=\tilde{P}$ constant (stationary filter)

v_1 and v_2 independent $\Rightarrow R_{12}=0$

$$\Rightarrow (1) \quad \hat{x}(k|k)=\hat{x}(k|k-1)+\tilde{K}(y(k)-C\hat{x}(k|k-1))$$

$$(2) \quad \hat{x}(k+1|k)=a\hat{x}(k|k-1)+bu(k)+K(y(k)-C\hat{x}(k|k-1))$$

where

$$\tilde{K}=\frac{cp}{c^2p+\sigma_e^2}, \quad K=\frac{acp}{c^2p+\sigma_e^2}$$

$$p=a^2p+\sigma_1^2-K^2(c^2p+\sigma_2^2)$$

$$(1) \quad =a^2p+\sigma_1^2-\frac{a^2c^2p^2}{c^2p+\sigma_2^2}=0.5p+\sigma_1^2-\frac{0.25p^2}{0.5p+\sigma_2^2} \quad (*)$$

$$0.5p^2+p\sigma_2^2=\cancel{0.25p^2}+0.5(\sigma_1^2+\sigma_2^2)p+\sigma_2^2\sigma_1^2-\cancel{0.25p^2}$$

$$0.5p^2+0.5p(\sigma_2^2-\sigma_1^2)=\sigma_2^2\sigma_1^2$$

$$\left(p+0.5(\sigma_2^2-\sigma_1^2)\right)^2=2\sigma_2^2\sigma_1^2+0.25(\sigma_2^4+\sigma_1^4+2\sigma_2^2\sigma_1^2)$$

$$=1.5\sigma_2^2\sigma_1^2+0.25(\sigma_2^4+\sigma_1^4)$$

$$p=0.5\left\{\sigma_1^2-\sigma_2^2\overset{+}{(-)}\sqrt{6\sigma_2^2\sigma_1^2+\sigma_2^4+\sigma_1^4}\right\}$$

$p>0$

$$\Rightarrow \tilde{K} = \frac{0.354\left(\sigma_1^2 - \sigma_2^2 + \sqrt{6\sigma_2^2\sigma_1^2 + \sigma_2^4 + \sigma_1^4}\right)}{0.25\left(\sigma_1^2 + \sqrt{6\sigma_2^2\sigma_1^2 + \sigma_2^4 + \sigma_1^4}\right) + 0.75\sigma_2^2}$$

$$K = \frac{\tilde{K}}{\sqrt{2}} = \ldots$$

For $\sigma_1 = \sigma_2 = 1 \Rightarrow P = \sqrt{2}$, $\tilde{K} = 0.59$, $K = \sqrt{2} - 1$

b) The variance of $x - \hat{x}(k|k)$ is (p 138 GL)

$$\tilde{P} = E\{\tilde{x}(k|k)^2\}$$

$$= (1 - \tilde{K}c)^2 P + \tilde{K}^2 R_2$$

$$= \ldots = 0.828$$

c) The pole of the Kalman filter follows from the system equation (2), i.e.

$$\lambda - (a - Kc) = 0$$

$$\lambda = \frac{1}{\sqrt{2}}\left(1 - \frac{\tilde{K}}{\sqrt{2}}\right)$$

$$\sigma_2^2 \gg \sigma_1^2 \Rightarrow$$

$$\tilde{K} \approx \frac{0.354\left(-\sigma_2^2 + \sqrt{6\sigma_2^2\sigma_1^2 + \sigma_2^4}\right) / \sigma_2^2}{\sqrt{6\sigma_2^2\sigma_1^2 + \sigma_2^4} \quad / \sigma_2^2}$$

$$= \frac{0.354\left(-1 + \sqrt{\frac{6\sigma_1^2}{\sigma_2^2} + 1}\right)}{\sqrt{\frac{6\sigma_1^2}{\sigma_2^2} + 1}} \rightarrow 0$$

when $\sigma_2^2 \rightarrow \infty$

Thus $K \to 0$ and the pole $\lambda \to a$
i.e. the same as the pole of the system model.

When $\sigma_2^2 \to \infty$ the measurement $y(k)$ becomes useless and the best estimate is simply given by running the model alone from u to $\hat{x}$.

(d)

L is given by the LQ solution to

$$\min \sum_{0}^{\infty} x^T(k)\, Q_1\, x(k) + u^T(k)\, Q_2\, u(k)$$

where $Q_1 = 1$ and $Q_2 = 2$

Glad & Ljung (9.37b) gives

$$S = 0.5 \cdot S + 1 - \frac{0.5 S^2}{S + 2}$$

$$0.5 S (S+2) - S - 2 + 0.5 S^2 = 0$$

$$S^2 = 2$$

$$S = (\pm)\sqrt{2} \qquad (S > 0 \text{ required})$$

(9.37a) gives

$$L = \frac{bSa}{2 + b^2 S} = \frac{1}{2 + \sqrt{2}} = \underline{\underline{0.29}}$$

(e)

$$\min \; Var(x - x_{ref}) + \alpha\, Var\{u\}$$

where $\alpha > 2$ (try for example 20)

(f)

Introduce integral state

$$x_I(k+1) = x_I(k) + \Big(x_{ref}(k) - \underbrace{\tfrac{1}{c}\, y(k)}_{\sim x(k)}\Big)$$

and add to feed back

$$u(k) = - L_x \hat{x}(k) - L_I\, x_I(k)$$

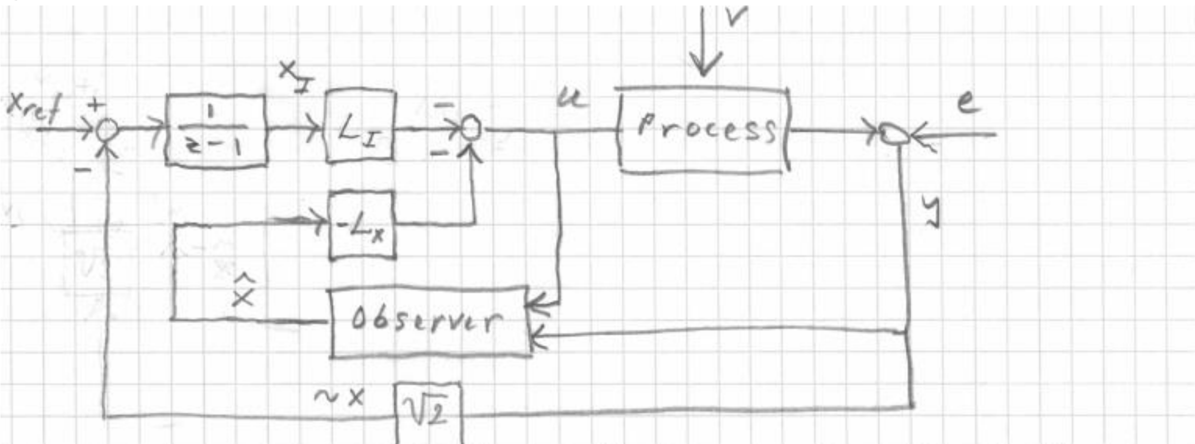
Extend model $\sqrt{2}\, y(k) = x(k) + \sqrt{2}\, e(k)$

$$\begin{bmatrix} x(k+1) \\ x_I(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ -1 & 1 \end{bmatrix}}_{A_e} \begin{bmatrix} x(k) \\ x_I(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B_e} u(k) + \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} v(k) \\ e(k) \end{bmatrix}$$

Solve $[L_x \; L_I] = LQ(A_e, B_e, Q_1, Q_2)$

with $Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$, $Q_2 = \beta$ (Eqn 9.37 again)

(g)



Note: Improved reference tracking can be achieved with reference feedforward

3.

a) $H_1(s) = \frac{2}{1+s} U(s)$, $H_2(s) = \frac{1}{1+2s} H_1(s)$

$(1+s) H_1(s) = 2U(s)$

$(1+2s) H_2(s) = H_1(s)$

Inverse Laplace transformation gives

$h_1(t) + \dot{h}_1(t) = 2u(t)$

$h_2(t) + 2\dot{h}_2(t) = h_1(t)$

with $x_1 = h_1$, $x_2 = h_2$ and $y = x_2$

$\dot{x}_1 = -x_1 + 2u$

$\dot{x}_2 = 0.5 x_1 - 0.5 x_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0.5 & -0.5 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 2 \\ 0 \end{bmatrix}}_{B} u$$

$$y = \underbrace{[0 \quad 1]}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) Observervability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.5 & -0.5 \end{bmatrix}$$

has linearly independent columns / rows

i.e. there is no α such that

$$\begin{bmatrix} 0 \\ 0.5 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} \Rightarrow \text{Observervable}$$

c)

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$
$$= (A - KC)\hat{x} + Bu + Ky$$

The observers poles are given by the eigenvalues of its system matrix

$$A - KC = \begin{bmatrix} -1 & 0 \\ 0.5 & -0.5 \end{bmatrix} - \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & -k_1 \\ 0.5 & -0.5 - k_2 \end{bmatrix}$$

Thus, the eigenvalues are given by

$$\det(\lambda I - [A - KC]) = \det\begin{pmatrix} \lambda + 1 & k_1 \\ -0.5 & \lambda + 0.5 + k_2 \end{pmatrix}$$
$$= (\lambda+1)(\lambda + 0.5 + k_2) - 0.5k_1$$
$$= \lambda^2 + (1.5 + k_2)\lambda + 0.5 + k_2 - 0.5k_1$$
$$= P(\lambda)$$

Double pole in -1 $\Leftrightarrow$ $P(\lambda) = (\lambda+1)^2$
$$= \lambda^2 + 2\lambda + 1$$

Polynomial identification gives

$$\begin{cases} 1.5 + k_2 = 2 \\ 0.5 + k_2 - 0.5k_1 = 1 \end{cases} \Rightarrow \begin{cases} k_2 = 0.5 \\ k_1 = 0 \end{cases}$$

$$K = [0 \quad 0.5]^T$$

4.

The colored disturbances can be regarded as outputs from a linear filter having white noise of intensity $R = 1$ as input.

$$w(t) = G_w(p)\, e_1(t) \quad , \quad e_1 \sim WN(0,1)$$
$$n(t) = G_n(p)\, e_2(t) \quad , \quad e_2 \sim WN(0,1)$$

Spectral factorization gives

$$\Phi_w(\omega) = |G_w(j\omega)|^2 = G_w(j\omega)\, G_w(-j\omega)$$
$$= \frac{2}{1+j\omega} \cdot \frac{2}{1-j\omega} = \frac{4}{\omega^2+1}$$

In the same way

$$\Phi_n(\omega) = |G_n(j\omega)|^2 = \frac{1+j\omega}{2+j\omega} \; \frac{1-j\omega}{2+j\omega} = \frac{1+\omega^2}{4+\omega^2}$$

$$\therefore \begin{cases} G_w(p) = \dfrac{2}{1+p} \\[2ex] G_n(p) = \dfrac{1+p}{2+p} = \dfrac{-1}{2+p} + 1 \end{cases}$$

direct term since degree of nominator = deg (denominator)

$$\Rightarrow \begin{cases} (1+p)\, w(t) = 2 e_1(t) \Rightarrow \dot{w} = -w + 2e_1 \\[2ex] n(t) = \underbrace{\dfrac{-1}{2+p}\, e_2(t)}_{x_n(t)} + e_2(t) \end{cases}$$

$$(2+p)\, x_n(t) = -v_2(t) \Rightarrow \dot{x}_n = -2x_n - e_2$$

With $x_1 = x$, $x_2 = w$, $x_3 = x_n$ we get

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x_e} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 2 & 0 \\ 0 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}}_{v_1}$$

$$y = [1 \quad 0 \quad 1]\, x_e + \underbrace{e_2(t)}_{v_2}$$

$$R = E\{v v^T\} = \begin{bmatrix} E\{v_1 v_1^T\} & E\{v_1 v_2^T\} \\ E\{v_2 v_1^T\} & E\{v_2 v_2^T\} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

5.

a) $$x(k+1) = \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}}_{A} x(k) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u(k) \qquad (1)$$

$$x(1) = A x(0) + B u(0)$$

$$x(2) = A x(1) + B u(1)$$
$$= A^2 x(0) + AB u(0) + B u(1)$$

$$x(2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{imply}$$

$$\begin{bmatrix} AB & B \end{bmatrix} \begin{bmatrix} u(0) \\ u(1) \end{bmatrix} = -A^2 x(0)$$

$$\begin{bmatrix} u(0) \\ u(1) \end{bmatrix} = \begin{bmatrix} AB & B \end{bmatrix}^{-1} A^2 x(0)$$
$$= \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} x(0) = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} x(0)$$

b) Repeating (1) n times and reorder gives

$$x(n) = A^n x(0) + B u(n-1) + AB u(n-2) + \dots + A^{n-1} B u(0)$$

$$\Rightarrow x(n) - A^n x(0) = \mathcal{O}(A,B) \begin{bmatrix} u(n-1) \\ \vdots \\ u(0) \end{bmatrix}$$

If this is to hold for any $x(0)$ and $x(n)$ the observability matrix $\mathcal{O}(A,B)$ must have full rank $\Leftrightarrow$ the system must be observable

A controller that can take the state to 0 in (less) than n samples is called a dead beat controller (all poles will be in 0)