CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Electrical Engineering
Division of Systems and Control

EXAMINATION IN LINEAR CONTROL SYSTEM DESIGN

(Course SSY285)

Monday April 14, 2025

Time and place:	08:30 - 14:30 (Johanneberg)
Teacher:	Torsten Wik (031 - 772 5146)

The total points achievable are 30. Unfinished solutions normally results in 0 points. Incorrect solutions with significant errors, unrealistic results that are not commented, or solutions that are too difficult to follow also result in 0 points.

Numerically incorrect calculations that do not cause unrealistic results will normally lead to reduction of 1 point. Incorrect answers that are consequences of a previous error and do not simplify the problem will not lead to any further reduction.

The scales for grading are

Grade 3: at least 12 points
Grade 4: at least 18 points
Grade 5: at least 24 points

The following aids are allowed:

1. Course text book *Control Theory* (or Swedish version *Reglerteori*) by T. Glad and L. Ljung and **one more** control textbook.

   Paper copies are accepted instead of books.
2. 1 piece of A4 paper, with *hand written* notes on both sides. Copied sheets are not allowed!
3. Mathematical and physical handbooks of tables, such as Physics handbook and Beta Mathematics Handbook.
4. Memory depleted, non-programmable pocket calculator.

Solutions to problems and exercises are not allowed in the notes!

Mobile telephones, laptops or tablets/iPads are not allowed!

Good Luck!

1. Decide if the following statements for linear time-invariant systems are true or not (short motivation is required):

   (a) For a controllable system we can always determine a deadbeat controller.

   (b) There is no unique state-space realization of a transfer function.

   (c) The condition number of the observability matrix increases from 7 to 7000 when a sensor is moved from one position to another. Can we expect better or worse estimations using a Kalman filter?

3 p.

2. Give a state-space realization of a system given by the following differential equation

$$M\frac{d^2}{dt^2}y(t) + F\frac{d}{dt}y(t) + Dy(t) = u(t)$$

   where M, F and D are constant koefficient matrices, and M is invertible.

2 p.

3. Assume we have derived a state space model [aug19,apr25]

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(k) \end{aligned}$$

   (a) Determine L in a time invariant state feedback $u(k) = -Lx(k) + Kr(k)$ such that both closed loop poles are 0.9.

3 p.

   (b) Determine K such that $y = r$ in steady state (provided we have a correct model and no disturbances).

1 p.

4. A system to be controlled is described by the following LTI-system:

$$\begin{aligned}\frac{d}{dt}x(t) &= \begin{bmatrix} -1 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t)\end{aligned}$$

(a) As can be seen from the state-space model, we only measure x_1 and x_2. Can we estimate x_3?

2 p.

(b) Show that the system is not controllable!

2 p.

(c) Using the Matlab command `[V,D]=eig(A)` we get the eigenvectors as columns in `V` and the corresponding eigenvalues in the diagonal of `D`, such that

$$A = \underbrace{\begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}}_{V^{-1}}$$

Why is the system unstable?

1 p.

(d) Diagonalize the system. Is the system stabilizable?

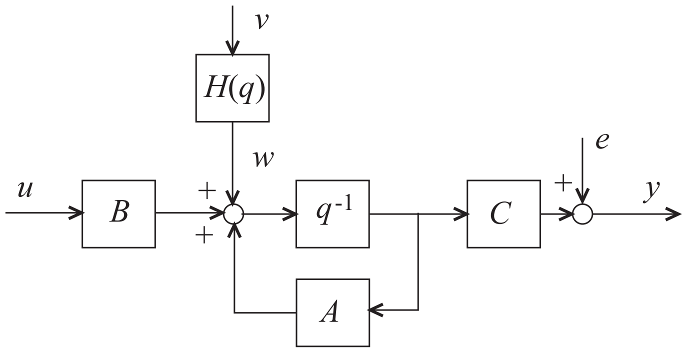
2 p.

5. Assume we have a system

$$x(k+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(k)$$
$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + e(k)$$

where e is a measurement white gaussian noise (WGN).

The process disturbance w is not white, though it originates from a source v that can be considered as WGN, see the figure below.



Neither w nor v can be measured online to be used in a feed forward control. However, from separate system identification experiments the pulse transfer operator H from v to w has been determined to be

$$H = \frac{\beta q^{-1}}{1 + \alpha_1 q^{-1} + \alpha_2 q^{-2}}$$

This can be used in an LQG controller by extending the state-space model and then estimate and feed back all states, i.e. the original process states and the disturbance states.

(a) Determine a state space model

$$x_w(k+1) = A_w x_w(k) + B_w v(k)$$
$$w(k) = C_w x_w(k)$$

corresponding to H.

2 p.

(b) Give an extended state-space model, including the disturbance states, on the correct form for design of a Kalman filter (you need not calculate the filter).

2 p.

6. A process variable x can be measured with two different methods. The measurement

$$y(k) = x(k) + v_{21}(k), \quad v_{21} \sim WGN(0,1),$$

where $WGN(0,1)$ denotes white normally (gaussian) distributed noise with zero mean and variance 1, gives an acceptable measurement variance.

A cheaper method gives the measurement

$$y(k) = x(k) + v_{22}(k), \quad v_{22} \sim WGN(0,2)$$

having twice the variance, which would be too high.

However, x is actually a dynamic process variable we can model by

$$x(k+1) = 0.7x(k) + u(k) + v_1(k), \quad v_1 \sim WGN(0,1)$$

where u is a known signal. Hence we can use a stationary Kalman filter to improve the estimate of x.

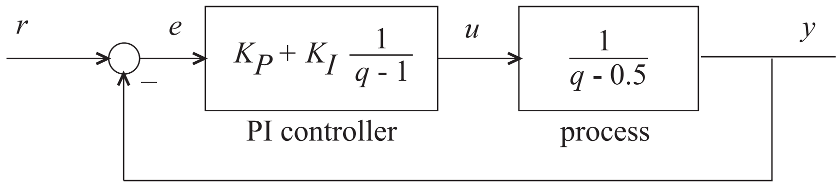
(a) What will the estimation error variance $E\{(x(k) - \hat{x}(k|k-1))^2\}$ be for the cheaper method?

3 p.

(b) Assume that we can have access to $y(k)$ when we make the estimate of $x(k)$, which should further improve the measurement. What will the estimation error variance $E\{(x(k) - \hat{x}(k|k))^2\}$ then be for the cheaper method? Is the accuracy sufficient?

2 p.

7. A discrete time first order process is to be controlled using a discrete time PI-controller. In the illustration below we use the time shift operator q (for example: $qy(k) \equiv y(k+1)$ and $y(k) = (1/(q-0.5)u(k))$.



The idea in this problem is to show how PI controller parameters can be determined by state feedback optimization.

(a) Formulate the process model on a standard state space form, i.e.

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}$$

1 p.

(b) Introduce the integral state

$$x_I(k) = \frac{1}{q-1} e(k)$$

and give an extended state space model

$$\begin{aligned} x_e(k+1) &= A_e x_e(k) + B_e u(k) + K_r r(k) \\ y(k) &= C_e x_e(k) \end{aligned}$$

including the integral state.

1 p.

(c) For the extended model we may now determine a state feedback

$$u(k) = -L_e x_e(k),$$

by minimization of (we may set $r = 0$, and note that $y^2 = x_e^T C_e C_e^T x_e$)

$$J = \sum_{k=0}^{\infty} y^2(k) + q_u u^2(k) + q_I x_I^2(k)$$

Give the necessary matrices and equations for finding the optimal L (you need not solve them)!

2 p.

Assume we have solved the Ricatti equations giving the optimal L_e. Express K_P and K_I in terms of L_e (you may assume $r = 0$).

1 p.