Solutions to SSY285 2025-04-14

1. (a) For a controllable system the poles of the feedback system (eigenvalues of A-BL) can be placed anywhere. A deadbeat controller is the controller that places all the poles in the origin. Consequently, this can be achieved for a controllable LTI system. This can also be shown mathematically using the controllability matrix.

   (b) Correct. This can be concluded from a simple example. The transfer function describes only the behavior from input to output. Let us assume we have a corresponding state-space model (A,B,C,D). If we divide B by 2 and multiply C by two, we get the same transfer function, $G(s) = 2C(sI-A)^{-1}B/2+D$.

   (c) When we are about to lose observability, the observability matrix is about to lose rank, which means that the condition number approaches infinity. If the condition number is low, state estimation is easy. If the condition number goes from 7 (very low) to 7000 (quite high) we can therefore expect the Kalman filter to produce worse estimations.

2. Let $x_1 = y$ and $x_2 = \dot{y}$. Then

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -M^{-1}Fx_2 - M^{-1}Dx_1 + M^{-1}u \end{aligned}$$

and thus

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & I \\ -M^{-1}D & -M^{-1}F \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u \\ y &= \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

3.

) $x(k+1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$

$u(k) = -Lx(k) + Kr(k) = -[l_1 \ \ l_2]x(k) + Kr(k)$

$\Rightarrow x(k+1) = \underbrace{\begin{bmatrix} 1-l_1 & -l_2 \\ 1 & 1 \end{bmatrix}}_{A_c} x(k) + \underbrace{\begin{bmatrix} K \\ 0 \end{bmatrix}}_{B_r} r(k)$

a) Poles $\equiv$ eigenvalues of A_c

$$\det(\lambda I - A_c) = \det \begin{bmatrix} \lambda - 1 + l_1 & l_2 \\ -1 & \lambda - 1 \end{bmatrix}$$

$$= (\lambda-1)(\lambda-1-l_1) + l_2$$

$$= \lambda^2 + \lambda(l_1 - 2) + l_2 - l_1 + 1$$

$$= (\lambda - 0.9)^2$$

$$= \lambda^2 - 1.8\lambda + 0.81$$

$\Rightarrow$ $\underline{\underline{l_1 = 0.2}}$ and $l_2 = 0.81 + l_1 - 1 = \underline{\underline{0.01}}$

b) $x(k+1) = A_c x(k) + B_r r(k)$

$y(k) = C x(k)$

Stable & SS $\Rightarrow x(k+1) = x(k) = \bar{x}$

$(I - A_c)\bar{x} = B_r \bar{r}$

$$\Rightarrow \bar{y} = C(I - A_c)^{-1} B_r \bar{r}$$

$$= [0 \ \ 1] \begin{bmatrix} l_1 & l_2 \\ -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} K \\ 0 \end{bmatrix} \bar{r}$$

$$= \underbrace{[0 \ \ 1] \frac{1}{l_2} \begin{bmatrix} 0 & -l_2 \\ 1 & l_1 \end{bmatrix} \begin{bmatrix} K \\ 0 \end{bmatrix}}_{=1} \bar{r}$$

$= 1 \Rightarrow 100K = 1 \Rightarrow \underline{\underline{K = 0.01}}$

4.

(a) The observability matrix is

$$O(A,C) = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -2 & 0 \\ 5 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix}$$

The first three rows are clearly linearly independent since subtracting the first row from the third gives a diagonal matrix with non-zero diagonal elements. Thus, the matrix has full rank (3), which means that it is observable and, consequently, the unmeasured state x_3 can be estimated.

(b) The controllability matrix is

$$S(A,B) = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 4 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

We see that the first, third and fifth column are equal (and thus linearly dependent!) and the second, fourth and last column are also linearly dependent, but not on the others. Thus, this matrix has rank 2, and one state (mode) is not observable.

(c) The matrix D has the eigenvalues on the diagonal. One of the eigenvalues is positive, i.e. in the right half-plane. The system is thus unstable.

(d) Let $z = Vx$. Then

$$\begin{aligned} \dot{z} &= V\dot{x} = VAx + VBu \\ &= VAV^{-1}z + VBu \\ &= \begin{bmatrix} -3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ \sqrt{2} & 0 \end{bmatrix} u \\ y &= Cx = CV^{-1}z \\ &= \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & -1 & 0 \end{bmatrix} z \end{aligned}$$

(e) From the diagonalization we see that it is the first state (z_1) that is not controllable. The eigenvalue for that state is the corresponding diagonal element of D, i.e. $-3 \in LHP$, and this state is therefore stable and will go to zero.

Since the only uncontrollable state is stable, the system is by definition stabilizable.

5.

a) $$w(k) = \frac{0.5q^{-1}}{(1+0.3q^{-1}+0.5q^{-2})} v(k)$$

$$w(k)+0.3w(k-1)+0.5w(k-2) = 0.5v(k-1)$$

$$w(k+1)+0.3w(k)+0.5w(k-1) = 0.5v(k)$$

$$\underbrace{\begin{bmatrix} w(k+1) \\ w(k) \end{bmatrix}}_{x_w(k+1)} = \begin{bmatrix} -0.3 & 0.5 \\ 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} w(k) \\ w(k-1) \end{bmatrix}}_{x_w(k)} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} v(k)$$

$$w(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} w(k) \\ w(k-1) \end{bmatrix}$$

b) $$\underbrace{\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ w(k+1) \\ w(k) \end{bmatrix}}_{x_e(k+1)} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -0.3 & 0.5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ w(k) \\ w(k-1) \end{bmatrix}}_{x_e(k)} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u(k)$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 0.5 \\ 0 \end{bmatrix} v(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x_e(k) + e(k)$$

6.

) $x(k+1) = 0.7x(k) + u(k) + v_1(k)$

$y(k) = x(k) + v_2(k)$

a) Prediction case

$$E\{(\hat{x}(k|k-1) - x(k))^2\} \equiv P(k|k-1)$$

where $P > 0$ is the solution to

$$P = APA^T + R_1 - (APC^T + R_{12})(C^TPC + R_2)^{-1}(CPA^T + R_{12}^T)$$

$$= 0.49P + 1 - 0.7P \frac{1}{P+2} 0.7P$$

$$P^2 + 2P - 0.49P^2 - 0.98P - P - 2 + 0.49P^2 = 0$$

$$1.49P^2 + 0.02P = 2$$

$$P^2 + 0.0134P = 1.342$$

$$(P + 0.0067)^2 = 1.342 + 0.0067^2 = 1.342$$

$$P = -0.0067 \underset{P>0}{(\pm)} \sqrt{1.342} = \underline{\underline{1.15}}$$

b) Filter case

$$P^*(k|k) \equiv E\{(\hat{x}(k|k) - x(k))^2\}$$

$$= P - PC^T(CPC^T + R_2)^{-1}CP$$

$$= 1.15 - \frac{1.15^2}{1.15+2} = \underline{\underline{0.73}}$$

which is less than the variance of method 1!

7.

a) $$y(k) = \frac{1}{q - 0.5} u(k) \Rightarrow y(k+1) - 0.5y(k) = u(k)$$

$\therefore$
$$x(k+1) = 0.5x(k) + u(k)$$
$$y(k) = x(k)$$

b) $$x_I(k) = \frac{1}{q-1}(r(k) - y(k))$$

$$\Rightarrow \quad x_I(k+1) - x_I(k) = r(k) - x(k)$$

$\therefore$
$$\underbrace{\begin{bmatrix} x(k+1) \\ x_I(k+1) \end{bmatrix}}_{x_e(k+1)} = \underbrace{\begin{bmatrix} 0.5 & 0 \\ -1 & 1 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x(k) \\ x_I(k) \end{bmatrix}}_{x_e(k)} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B_e} u(k) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{K_r} r(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_e} x_e(k)$$

c) Optimal control law (Assume $r = 0$ and $L_e = [l_1\ l_2]$)

$$u(k) = -L_e x_e(k)$$
$$= -l_1 x(k) - l_2 x_I(k)$$
$$= -l_1 y(k) + l_2 \frac{1}{q-1} y(k)$$

Block scheme gives

$$u(k) = -\left(K_p + K_I \frac{1}{q-1}\right) y(k)$$

$\therefore \quad K_p = l_1 \quad \text{and} \quad K_I = -l_2$