CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Electrical Engineering
Division of Systems and Control

# EXAMINATION IN LINEAR CONTROL SYSTEM DESIGN

(Course SSY285)

Monday January 13, 2025

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| Time and place: | 14:00 - 18:00 (Johanneberg) |
| Teacher: | Torsten Wik (031 - 772 5146) |

The total points achievable are 30. Unfinished solutions normally results in 0 points. Incorrect solutions with significant errors, unrealistic results that are not commented, or solutions that are too difficult to follow also result in 0 points.

Numerically incorrect calculations that do not cause unrealistic results will normally lead to reduction of 1 point. Incorrect answers that are consequences of a previous error and do not simplify the problem will not lead to any further reduction.

The scales for grading are

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| Grade 3: | at least 12 points |
| Grade 4: | at least 18 points |
| Grade 5: | at least 24 points |

The following aids are allowed:

1. Course text book *Control Theory* (or Swedish version *Reglerteori*) by T. Glad and L. Ljung and **one more** control textbook.

   Paper copies are accepted instead of books.

2. 1 piece of A4 paper, with *hand written* notes on both sides. Copied sheets are not allowed!

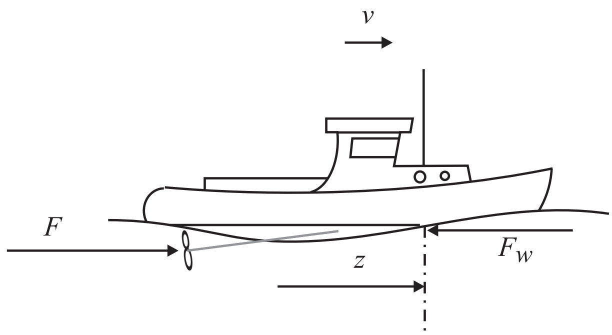
3. Mathematical and physical handbooks of tables, such as Physics handbook and Beta Mathematics Handbook.

4. Memory depleted, non-programmable pocket calculator.

**Solutions to problems and exercises are not allowed in the notes!**

**Mobile telephones, laptops or tablets/iPads are not allowed!**

1. A fishing boat is to be equipped with a speed and position over ground (bottom) control based on GPS position. Here we may assume only one space dimension, i.e. the boat follows a straight line (see figure) and the position is $x$.




Let the boat position be $z(t)$ $[m]$, the boat speed be $v(t)$ $[m/s]$ and the propulsion force be $F(t)$ $[N]$. We may assume all friction forces together be related to the speed according to

$$F_w(t) = 400v^2(t) \quad [N]$$

The boat mass is 10 000 kg.

(a) Determine a continuous time state space model for the boat propulsion, describing the behavior from propulsion force $u = F$ to position and speed $y = \begin{bmatrix} z & v \end{bmatrix}^T$.

2 p.

(b) The normal speed is 5 m/s (approximately 10 knots). Show that the linear state space model describing the dynamic behavior around that speed is

$$\begin{aligned} \frac{d}{dt}\Delta x(t) &= \begin{bmatrix} 0 & 1 \\ 0 & -0.4 \end{bmatrix} \Delta x(t) + \begin{bmatrix} 0 \\ 10^{-4} \end{bmatrix} \Delta u(t) \\ \Delta y(t) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) \end{aligned}$$

2 p.

(c) Discretize the above state space model for zero order hold and a sampling time of 10 ms.

2 p.

2. **Note: Many of the subproblems can be solved independently of eachother!**

Consider the following linear time discrete system with two separate measurements of a scalar state variable:

$$\begin{aligned} x(t+1) &= 0.5x(t) + u(t) + v_1(t) \\ y(t) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} x(t) + v_2(t) \end{aligned}$$

where $v_1$ and $v_2$ are uncorrelated white noise with 0 average and variances

$$E\{v_1^2\} = 1 \qquad \text{and} \qquad E\{v_2 v_2^T\} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

(a) The state $x(t)$ is estimated using a stationary Kalman filter:

$$\hat{x}(t+1|t) = A\hat{x}(t|t-1) + Bu(t) + K(y(t) - C\hat{x}(t|t-1))$$

Give $A$, $B$, $C$, calculate $K$ and determine the stationary variance of $\hat{x}(t) = \hat{x}(t|t-1)$.

3 p.

(b) The sampling rate in the control system is low enough to actually use the latest measurement $y(t) = \begin{bmatrix} y_1(t) & y_2(t) \end{bmatrix}$ to estimate $x(t)$.
What is the stationary variance of the estimate $\hat{x}(t) = \hat{x}(t|t)$ then?

3 p.

(c) We may express our observer on the form

$$\hat{x}(t|t) = H_{y1}(q)y_1(t) + H_{y2}(q)y_2(t) + H_u(q)u(t)$$

where $H_{y1}(q)$, $H_{y2}(q)$ and $H_u(q)$ are transfer operators (transfer functions if the shift operator $q$ is replaced by $z$). Determine $H_{y1}$, $H_{y2}$ and $H_u$.

2 p.

(d) We wish to minimize the following weighted variance by state feedback:

$$V = E\{x^2(t)\} + Q_u E\{u^2(t)\}$$

Express the corresponding optimal controller on the form

$$u(t) = F_{y1}(q)y_1(t) + F_{y2}(q)y_2(t)$$

If you did not manage to determine the transfer operators in (c) you may assume

$$H_{y1}(q) = \frac{\beta_1}{1+\alpha q^{-1}}, \quad H_{y2}(q) = \frac{\beta_2}{1+\alpha q^{-1}}, \quad \text{and} \quad H_u(q) = \frac{\gamma}{1+\alpha q^{-1}}.$$

4 p.

3. A continuous time linear system is given by

$$y(t) = \begin{bmatrix} \frac{1}{s} & 2 \\ 0 & \frac{1}{s} \\ \frac{1}{s+1} & \frac{2}{s+1} \end{bmatrix} u(t)$$

(a) What are the poles and zeros of this system?

2 p.

(b) Derive a state space realization (model) of the system. Is your model a minimal realization?

2 p.

4. The following state-space model has been derived for a system

$$\begin{aligned} \frac{d}{dt}x(t) &= \begin{bmatrix} -1 & 0 \\ 0.5 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + v_1(t) \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + v_2(t) \end{aligned}$$

where we assume the disturbances are two independent white gaussian disturbances with variances

$$R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad R_2 = 1$$

Based on this a controller is to be designed to minimize

$$V = \int_0^\infty x^T(t)Q_x x(t) + u(t)^T Q_u u(t) dt$$

where $Q_x$ and $Q_u$ are positive definite.

(a) Show why this is possible, i.e. that all necessary conditions holds.

2 p.

(b) We want to also control the system such that the output $y$ on average is close to a setpoint that is changed now and then. Because the disturbances are not necessarily zero mean this means that we need to add integral action. State all the matrices and equations needed to calculate the controller. You do not need to solve the equations, just state them. How do you change the tuning such that you get more integral action?

4 p.

(c) It also turns out that the disturbance $v_2$ is not white. Through separate measurements one have determined the spectrum of this disturbance to be

$$\Phi_2(\omega) = \frac{3}{1 + 4\omega^2}$$

How does this affect the equations you need to solve to calculate your controller?

2 p.

Good luck!