Solutions to SSY285 2025-01-13

1.

a) Newtons law: $ma = \sum F$

$\Rightarrow m\ddot{z} = F - F_w = u - 400 v^2$

Let $x_1 = z$ and $x_2 = v = \dot{z}$. This gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{m}(u - 400 x_2^2) \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) Let $\Delta x(t) = x(t) - \bar{x}$, $\Delta u(t) = u(t) - \bar{u}$, and $\Delta y(t) = y(t) - \bar{y}$, where $\bar{x}, \bar{u}, \bar{y}$ is the equilibrium operating point where the speed $\bar{v} = \bar{x}_2 = 5$ m/s. The linearized model then becomes

$$\frac{d}{dt}\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} = \begin{bmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} \end{bmatrix}_{(\bar{x},\bar{u})} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} \frac{df_1}{du} \\ \frac{df_2}{du} \end{bmatrix}_{(\bar{x},\bar{u})} \Delta u$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & \frac{800\bar{x}_2}{m} \end{bmatrix} \Delta x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \Delta u$$

$$= \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0{,}4 \end{bmatrix}}_{A} \Delta x + \underbrace{\begin{bmatrix} 0 \\ 10^{-4} \end{bmatrix}}_{B} \Delta u$$

$$\Delta y = \Delta x = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \Delta x$$

c) Discretization 'zoh' with $h = 0.01\,s$:

$$\Delta x(k+1) = A_d\, \Delta x(k) + B_d\, \Delta u(k)$$
$$\Delta y(k) = C\, \Delta x(k)$$

$$A_d = e^{Ah} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}_{t=h}$$

$$(sI-A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s+0.4 \end{bmatrix}^{-1} = \frac{1}{s(s+0.4)} \begin{bmatrix} s+0.4 & 1 \\ 0 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+0.4)} \\ 0 & \frac{1}{s+0.4} \end{bmatrix}$$

$$\mathcal{L}^{-1}\{(sI-A)^{-1}\}_{t=h} = \begin{bmatrix} 1 & 2.5(1-e^{-0.4h}) \\ 0 & e^{-0.4h} \end{bmatrix}$$

$$h = 0.01 \quad \Rightarrow \quad \underline{\underline{A_d = \begin{bmatrix} 1 & 0.01 \\ 0 & 0.996 \end{bmatrix}}}$$

$$B_d = \int_0^h e^{At} B\, dt = \int_0^h \begin{bmatrix} 2.5\cdot 10^{-4}(1-e^{-0.4t}) \\ 10^{-4}\cdot e^{-0.4t} \end{bmatrix} dt$$

$$= \begin{bmatrix} 2.5\cdot 10^{-4}(h + 2.5(e^{-0.4h}-1)) \\ -2.5\cdot 10^{-4}(e^{-0.4h}-1) \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} 5.0\cdot 10^{-9} \\ 1.0\cdot 10^{-6} \end{bmatrix}}}$$

2.

a) We have $A = 0.5$ $\quad R_1 = 1$

$B = 1$ $\quad R_{12} = [0 \ 0]$ (v_1 & v_2 uncorr)

$C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Kalman filter (predictor case):

$$\hat{x}(t+1/t) = A\hat{x}(t/t-1) + Bu(t) + \bar{K}(y(t) - C\hat{x}(t/t-1))$$

$$\bar{K} = A\bar{P}C^T(R_2 + C\bar{P}C^T)^{-1}$$

$$= 0.5\bar{P}[1 \ 1]\begin{bmatrix} 1+\bar{P} & \bar{P} \\ \bar{P} & 2+\bar{P} \end{bmatrix}^{-1} =$$

$$= 0.5\bar{P}[1 \ 1]\frac{1}{3\bar{P}+2}\begin{bmatrix} 2+\bar{P} & -\bar{P} \\ -\bar{P} & 1+\bar{P} \end{bmatrix}$$

$$= \frac{0.5\bar{P}}{3\bar{P}+2}[2 \ 1]$$

$$\bar{P} = A\bar{P}A^T + R_1 - KC\bar{P}\Phi^T$$

$$= 0.25\bar{P} + 1 - \frac{0.5\bar{P}}{3\bar{P}+2}[2 \ 1]\begin{bmatrix} 1 \\ 1 \end{bmatrix}\bar{P}\cdot 0.5$$

$$= 0.25\bar{P} + 1 - \frac{0.75\bar{P}^2}{3\bar{P}+2}$$

$$0.75\bar{P}(3\bar{P}+2) + 0.75\bar{P}^2 = 3\bar{P}+2$$

$$3\bar{P}^2 - 1.5\bar{P} = 2$$

$$\bar{P}^2 - 0.5\bar{P} = 0.667$$

$$(\bar{P} - 0.25)^2 = 0.667 + 0.0625 = 0.7292$$

$$Var\{\hat{x}(t/t-1)\} \equiv \bar{P} = 0.25 (\pm)\sqrt{0.7292} = \underline{\underline{1.104}}$$

$$\Rightarrow \quad \bar{K} = [0.208 \quad 0.104]$$

16) Stationary Kalman (filter case)

$$\hat{x}(t|t) = \hat{x}(t|t-1) + \bar{K}_f\left(y(t) - C\hat{x}(t|t-1)\right)$$

$$= \left(I - \bar{K}_f C\right)\left(A\hat{x}(t-1|t-1) + Bu(t-1)\right) + \bar{K}_f y(t)$$

where

$$\bar{K}_f = \bar{P}C^T\left(C\bar{P}C^T + R_2\right)^{-1} = \frac{\bar{K}}{0.5} = [0.416 \quad 0.208]$$

$$\mathrm{Var}\{\hat{\tilde{x}}(t|t)\} \equiv \bar{P}(t|t)$$

$$= \bar{P} - \bar{K}_f C\bar{P}$$

$$= 1.104\,(1 - 0.416 - 0.208) = \underline{\underline{0.415}}$$

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c) $$\hat{x}(t) = \left(1 - [0.416 \ \ 0.208]\begin{bmatrix}1\\1\end{bmatrix}\right)\left(0.5\hat{x}(t-1) + u(t-1)\right)$$

$$+ [0.416 \quad 0.208]\begin{bmatrix}y_1(t)\\y_2(t)\end{bmatrix}$$

$$= 0.188\,\hat{x}(t-1) + 0.376\,u(t)$$

$$+ 0.416\,y_1(t) + 0.208\,y_2(t)$$

$$\left(1 - 0.188\,q^{-1}\right)\hat{x}(t) = 0.38\,u(t) + 0.42\,y_1(t) + 0.21\,y_2(t)$$

$$\Rightarrow \hat{x}(t) = \hat{x}(t|t) = \underbrace{\frac{0.38}{1 - 0.19q^{-1}}}_{H_u} u(t) + \underbrace{\frac{0.42}{1 - 0.19q^{-1}}}_{H_{y_1}} y_1(t)$$

$$+ \underbrace{\frac{0.21}{1 - 0.19q^{-1}}}_{H_{y_2}} y_2(t)$$

d) $u(t) = -Lx(t)$

where

$$L = \arg\min_{L} \underbrace{E\{x^2(t)\} + Q_u E\{u^2(t)\}}_{\lim_{N\to\infty} \frac{1}{N}\sum_0^N x^2(t) + Q_u u^2(t)}$$

$$= \arg\min_{L} \sum_0^N x^2(t) + Q_u \sum_0^N u^2(t)$$

$$= \arg\min \|x\|^2_{Q_1} + \|u\|^2_{Q_2}, \quad Q_1 = 1, \; Q_2 = Q_u$$

$$L = (B^T S B + Q_2)^{-1} B^T S A \qquad (9.37a)$$

$$= \frac{0,5S}{S + Q_u}$$

$$S = A^T S A + Q_1 - A^T S B L \qquad (9.37b)$$

$$= 0.25S + 1 - \frac{0.25S^2}{S + Q_u}$$

$$0.75S(S + Q_u) + 0,25S^2 = S + Q_u$$

$$S^2 + S\underbrace{(0.75Q_u - 1)}_{a} = Q_u$$

$$\left(S + \frac{a}{2}\right)^2 = Q_u + \frac{a^2}{4}$$

$$S = -\frac{a}{2} \overset{+}{(-)} \sqrt{Q_u + \frac{a^2}{4}}$$

$$= \frac{1}{2}\left(1 - 0.75Q_u + \sqrt{4Q_u + \frac{9}{16}Q_u^2 + 1 - \frac{3}{4}Q_u}\right)$$

$$= \frac{1}{2}\left(1 - 0.75Q_u + \sqrt{13Q_u + \frac{9}{16}Q_u^2 + 1}\right)$$

$$\equiv \bar{S}$$

$$L = \frac{0.5\bar{S}}{\bar{S} + Q_u} \equiv \bar{L}$$

$$u(t) = -\bar{L}\hat{x}(t) =$$

$$= -\frac{\bar{L}}{1+\alpha q^{-1}}\left(\beta_1 y_1(t) + \beta_2 y_2(t) + \gamma u(t-1)\right)$$

$$\left(1 + \frac{\bar{L}\gamma q^{-1}}{1+\alpha q^{-1}}\right) u(t) = \frac{-\bar{L}}{1+\alpha q^{-1}}\left(\beta_1 y_1(t) + \beta_2 y_2(t)\right)$$

$$\left(1 + \alpha q^{-1} + \bar{L}\gamma q^{-1}\right) u(t) = -\bar{L}\beta_1 y_1(t) - \bar{L}\beta_2 y_2(t)$$

$$\Rightarrow \quad u(t) = \underbrace{\frac{-\bar{L}\beta_1}{1+(\alpha+\bar{L}\gamma)q^{-1}}}_{F_{y_1}(q)} y_1(t) + \underbrace{\frac{-\bar{L}\beta_2}{1+(\alpha+\bar{L}\gamma)q^{-1}}}_{F_{y_2}(q)} y_2(t)$$

3.

a)

$$G(s) = \begin{bmatrix} 1/s & 2 \\ 0 & 1/s \\ 1/s+1 & 2/s+1 \end{bmatrix}$$

The largest minors (order 2) are

$$1/s^2, \quad \frac{2}{s(s+1)} - \frac{2}{s+1} = \frac{2(1-s)}{s(s+1)}, \quad \frac{-1}{s(s+1)}$$

Minors of order 1 are the elements of G

The pole polynomial is the least common divisor of all minors, i.e.

$$P(s) = s^2(s+1) \Rightarrow \begin{cases} 2 \text{ poles in } 0 \\ 1 \text{ pole in } -1 \end{cases}$$

The largest minors on common divisor are

$$\frac{s+1}{P(s)}, \quad \frac{2s(1-s)}{P(s)}, \quad \frac{-s}{P(s)}$$

The zero polynomial is the largest common divisor, which is 1 in this case

$\Rightarrow$ No zeros

b) $y_1 = \frac{1}{s} u_1 + 2u_2$

$y_2 = \frac{1}{s} u_2$

$y_3 = \frac{1}{s+1} u_1 + \frac{2}{s+1} u_2$

Let, for example,

$x_1 = \frac{1}{p} u_1 \;,\; x_2 = \frac{1}{p} u_2 \;,\; x_3 = \frac{1}{p+1} u_1 \;,\; x_4 = \frac{1}{p+1} u_2$

This gives

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

System has 3 poles and, hence, there is a statespace model with only 3 states that has the same input-output behavior.

Thus, this is not a minimal representation

4.

a) $\begin{cases} \dot{x} = Ax + Bu + v_1 \\ y = Cx + v_2 \end{cases}$

$$A = \begin{bmatrix} -1 & 0 \\ 0.5 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Minimizing

$$V = \int_0^\infty x^T Q_x x + u^T Q_u u \, dt$$

is an LQ problem, but since we do not have access to the states, we need to estimate them. The optimal solution is then to combine LQ and Kalman.

According to Th. 9.1, in addition to $Q_x \geq 0$ and $Q_u > 0$ we have that

* (A, B) stabilizable

The eigenvalues of A are $\lambda_{1,2} = -1 \in LHP$

$\Rightarrow$ stable even without control

Alternatively

$\mathcal{S}(A,B) = [B \ AB] = \begin{bmatrix} 1 & -1 \\ 0 & 0.5 \end{bmatrix}$ has full rank

$\Rightarrow$ controllable $\Rightarrow$ stabilizable

* (A, Q_x) detectable

$$\mathcal{O}(A, Q_x) = \begin{bmatrix} Q_x \\ Q_x A \end{bmatrix}$$

$Q_x > 0 \Rightarrow$ full rank

(A, Q_x) observable $\Rightarrow$ detectable

5.

* $\tilde{R}_1 = R_1 - R_{12} R_2^{-1} R_{12}^T \geq 0$

$R_{12} = [0 \quad 0]^T \Rightarrow \tilde{R}_1 = R_1 > 0$

* (A, C) detectable

$$\mathcal{O}(A,C) = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0.5 & -1 \end{bmatrix} \text{ full rank}$$

$\Rightarrow$ detectable

* $(\underbrace{A - R_{12} R_2^{-1} C}_{A}, \underbrace{\tilde{R}_1}_{R_1})$ stabilizable

$S(A, R_1) = [R_1 \quad AR_1]$

$R_1 > 0 \Rightarrow$ full rank $\Rightarrow S$ full rank

$\Rightarrow$ controllable $\Rightarrow$ stabilizable

$\therefore$ All conditions for $\exists$ optimal controller satisfied

The controller

b) Extend the given model with an integral state

$$x_I = \int_0^t (r(\tau) - y(\tau))\,d\tau$$

$$\Rightarrow \dot{x}_I = r - Cx - v_2$$

The extended state-space model becomes

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix}}_{\dot{x}_e} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0.5 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x \\ x_I \end{bmatrix}}_{x_e} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B_e} u + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B_r} r + \underbrace{\begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}}_{V_e}$$

$$y = \underbrace{[\,0 \quad 1 \quad 0\,]}_{C_e} x_e + v_2$$

The optimal controller is now given by LQR solution applied to the extended model, i.e.

$$u(t) = -L_e \hat{x}_e(t)$$

$$\dot{\hat{x}}_e(t) = A_e \hat{x}(t) + B_e u(t) + B_r r(t) + K(y(t) - C\hat{x}(t))$$

where $P > 0$ solution to

$$A_e P + P A_e^T + R_{1e} - (P C_e^T + R_{12e}) R_2^{-1} (P C_e^T + R_{12e})^T = 0$$

$$K = (P C_e^T + R_{12}) R_2^{-1}$$

and $S > 0$ solution to

$$L_e = -Q_2^{-1} B_e^T S$$

$$A_e^T S + S A + Q_1 - S B Q_2^{-1} B_e^T S = 0$$

The cost matrix for the states can be set to be

$$Q_1 = \begin{bmatrix} Q_x & 0 \\ 0 & Q_I \end{bmatrix}$$

where more integral action is achieved by increasing Q_I

The variance matrix R_{1e} and R_{12e} are given by

$$Var\left\{\begin{bmatrix} v_e \\ v_2 \end{bmatrix}\right\} = E\left\{\begin{bmatrix} v_1 \\ -v_2 \\ v_2 \end{bmatrix}\begin{bmatrix} v_1^T & -v_2 & v_2 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} R_1 & \cancel{-R_{12}} & \cancel{R_{12}} \\ \cancel{-R_{12}^T} & R_2 & -R_2 \\ \cancel{R_{12}^T} & -R_2 & R_2 \end{bmatrix} = \begin{bmatrix} R_{1e} & R_{12e} \\ R_{12e}^T & R_2 \end{bmatrix}$$

$$R_{1e} = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}, \quad R_{12e} = \begin{bmatrix} 0 \\ -R_2 \end{bmatrix}$$

c) If the spectrum of v_2 is

$$\Phi_2(\omega) = \frac{3}{1+4\omega^2}$$

then the spectral theorem gives that

$$v_1(t) = G_1(p)\, v(t) \quad , \quad v \sim WGN(0,1)$$

where

$$\Phi_2(\omega) = G(j\omega)\, G(-j\omega)$$

Thus $\quad G(p) = \dfrac{\sqrt{3}}{1+2p} \quad$ and

$$(1+2p)\, v_2(t) = \sqrt{3}\, v(t)$$

$$\dot{v}_2(t) = -0.5\, v_2(t) + \frac{\sqrt{3}}{2} v(t)$$

This becomes yet another state, $x_v = v_1$, to add to the existing ones

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{x}_I \\ \dot{x}_v \end{bmatrix}}_{\dot{x}_e} = \underbrace{\begin{bmatrix} A & 0 & 0 \\ -C & 0 & -1 \\ 0 & 0 & -0.5 \end{bmatrix}}_{A_e} \underbrace{\begin{bmatrix} x \\ x_I \\ x_v \end{bmatrix}}_{x_e} + \underbrace{\begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}}_{B_e} u + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} r + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{N} \begin{bmatrix} v_1 \\ v \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix} x_e$$

The same equations as before need to be solved but with

$$Q_1 = \begin{bmatrix} Q_x & 0 & 0 \\ 0 & Q_I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R_{1e} = N \begin{bmatrix} R_1 & 0 \\ 0 & 1 \end{bmatrix} N^T, \quad R_{12e} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$