Modelling and Simulation ESS101
29 October 2025, Final Exam

This exam contains 10 pages (including this cover page) and 4 problems.

You are allowed to use the following material:

- *Modelling And Simulation, Lecture notes for the Chalmers course ESS101*, by S. Gros (Annotations are not allowed. Colored highlights and tabs/bookmarks for quick browsing are allowed.)
- *Mathematics Handbook* (Beta)
- *Physics Handbook*
- Chalmers approved calculator
- Formula sheet, appended to the exam.

– Organize your work in a reasonably neat and coherent way. Work scattered all over the page without a clear ordering may receive less credit.

– Mysterious or unsupported answers will not receive credit, but an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

– None of the proposed questions require extremely long computations. If you get caught in endless algebra, you have probably missed the simple way of doing it.

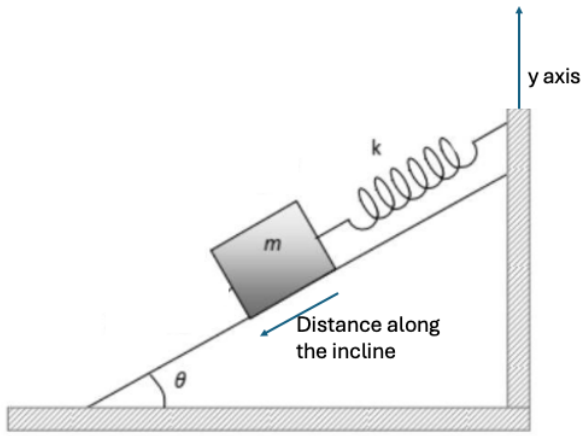
– The passing grade will be given at 20 points, grade 4 at 27 and the top grade at 34 points.

Problem	Points	Score
1	12	
2	9	
3	7	
4	12	
Total:	40	

Best of luck to all !!

TA: Ahmet Tekden, +46 31 772 50 91
TA: Filip Rydin, +46 72 560 00 08
Examiner: Yasemin Bekiroglu, +46 70 148 72 71

1. (a) (3 points) Consider the system depicted in the following figure where a mass m slides on an inclined frictionless surface with a spring. The spring stiffness k contributes to potential energy with the term $\frac{1}{2}kx^2$. Gravity is the only force that acts on the sytem. Let x be the distance measured along the incline, and use the usual term with veloctiy in your Kinetic energy formula. Write down the corresponding Lagrange function using generalized coordinates.



(b) (3 points) Write Euler Lagrange equations that describe the dynamics of this system.

(c) (3 points) What is the index of the following DAE? (All variables are time dependent.)

$\dot{\mathbf{x}}_1 = x_2$
$\dot{\mathbf{x}}_2 = z$
$0 = \frac{1}{2}(x_1^2 + x_2^2 - 2z)$

(d) (3 points) What is the index of the following DAE? (All variables are time dependent.)

$\dot{\mathbf{x}}_1 = x_2$
$\dot{\mathbf{x}}_2 = z$
$0 = \frac{1}{2}(x_1^2 + x_2^2 - 1)$

Solution:

$q = x$ (x along the incline)

$T = \frac{1}{2} m \dot{x}^2$

$U = \frac{1}{2} k x^2 - m g x \sin\theta$

$L = T - U$

$$L(q, \dot{q}) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 + m g x \sin\theta$$

$$\frac{d}{dt} \nabla_{\dot{q}} L - \nabla_q L = 0$$

$$m\ddot{x} - (-kx + mg\sin\theta) = 0$$

$$m\ddot{x} + kx - mg\sin\theta = 0$$

$$m\ddot{x} = mg\sin\theta - kx$$

other solutions will be also accepted as long as you describe your choice of q and follow the steps. If you introduce a constraint make sure you incorporate that as we did in the lectures.

(a)

all variables are time dependent

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = z$$

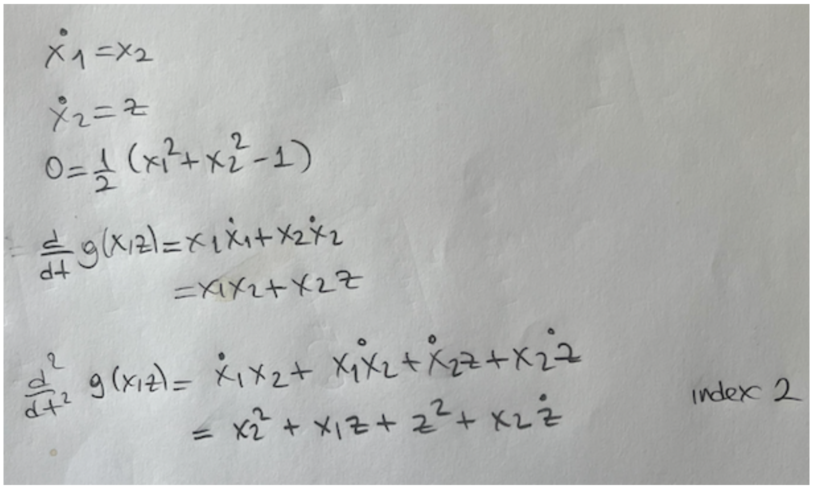
$$0 = \frac{1}{2}(x_1^2 + x_2^2 - 2z)$$

$$\frac{d}{dt} g(x, z) = x_1 \cdot \dot{x}_1 + x_2 \dot{x}_2 - \dot{z} = 0$$

$$0 = x_1 x_2 + x_2 z - \dot{z}$$

index 1

(b)



(c)

2. (a) (3 points) Given the following set of data points: $x_i, y_i = (1,2),(2,3),(3,5),(4,4),(5,6)$ Find the parameters of the line that best fits the data using the linear least squares method. (Hint: Based on F_N, R_N matrices)

   (b) (3 points) Consider the model $y(t) + ay(t-1) = bu(t-1) + e(t)$ where u is input, show how to calculate the parameters a and b using linear least squares method. (Hint: Based on F_N, R_N matrices)

   (c) (3 points) Consider the following systems, which model do they correspond to? Also write their predictors.
   $y(t) - 0.5u(t-1) = 0.2u(t-2) + e(t)$
   $y(t) - 0.7y(t-1) = 0.5u(t-1) + e(t)$
   $y(t) - 0.5y(t-1) = 0.3u(t-1) + e(t) + 0.4e(t-1)$

Solution:

(a) $\sum_{i=1:5} x_i = 1+2+3+4+5 = 15$, $\sum y_i = 2+3+5+4+6 = 20$,
$\sum {x_i}^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$, $\sum x_i y_i = 1x2 + 2x3 + 3x5 + 4x4 + 5x6 = 69$
$[a \quad b]^T = {R_N}^{-1} F_N$

$$R_N = \begin{bmatrix} 1 & 15/5 \\ 15/5 & 55/5 \end{bmatrix}$$

,

$$F_N = \begin{bmatrix} 20/5 \\ 69/5 \end{bmatrix}$$

which yields $a = 1.3b = 0.9$.

(b)

$$\begin{bmatrix} \hat{a_N} \\ \hat{b_N} \end{bmatrix} = \left(\frac{1}{N} \begin{bmatrix} \sum y^2(t-1) & -\sum y(t-1)u(t-1) \\ -\sum y(t-1)u(t-1) & \sum u^2(t-1) \end{bmatrix} \right)^{-1} \left(\frac{1}{N} \begin{bmatrix} -\sum y(t)y(t-1) \\ \sum y(t)u(t-1) \end{bmatrix} \right)$$

(c) $y(t) - 0.5u(t-1) = 0.2u(t-2) + e(t)$ FIR, depends on past inputs.

> $\hat{\mathbf{y}}(t|t-1) = 0.5u(t-1) + 0.2u(t-2)$
>
> $y(t) - 0.7y(t-1) = 0.5u(t-1) + e(t)$ ARX, depends on past inputs and past outputs.
>
> $\hat{\mathbf{y}}(t|t-1) = 0.7y(t-1) + 0.5u(t-1)$
>
> $y(t) - 0.5y(t-1) = 0.3u(t-1) + e(t) + 0.4e(t-1)$ ARMAX, depends on past inputs, past outputs and past noise.
>
> $\hat{\mathbf{y}}(t|t-1) = 0.5y(t-1) + 0.3u(t-1) + 0.4(y(t-1) - \hat{\mathbf{y}}(t-1|t-2))$

3. (a) (4 points) Apply the Newton method to solve the following system

$$xe^y = 1,$$
$$-x^2 + y = 1.$$

Use the initial guess $[x_0, y_0] = [0, 0]$. Calculate the resulting solution from applying Newton iteration only once (using full step), i.e. $[x_1, y_1]$.

(b) (3 points) Consider the function $f(x) = x^4 - 3x^3 + 2$, use the Newton method to optimize this function, namely to find the stationary points (minima or maxima) of f(x), starting from the initial guess $x_0 = 0.5$, and performing one iteration. What would x_1 be?

> **Solution:**
>
> (a) $f(x, y) = \begin{bmatrix} xe^y - 1 \\ -x^2 + y - 1 \end{bmatrix}$
>
> Plugging in the following in the update formula: $\partial f(x, y) = \begin{bmatrix} e^y & xe^y \\ -2x & 1 \end{bmatrix}$,
>
> $det\partial f(x, y) = e^y + 2x^2e^y$, $\partial f(x, y)^{-1} = \frac{1}{det\partial f(x,y)} \begin{bmatrix} 1 & -xe^y \\ 2x & e^y \end{bmatrix} = \frac{1}{1+2x^2} \begin{bmatrix} e^{-y} & -x \\ 2xe^{-y} & 1 \end{bmatrix}$
>
> Thus, $\begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} x_{k-1} \\ y_{k-1} \end{bmatrix} - \partial f(x_{k-1}, y_{k-1})^{-1}.f(x_{k-1}, y_{k-1})$
>
> Beginning with $x_0 = 0, y_0 = 0$, $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \frac{1}{1+0^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, which yields $x_1 = 1$, $y_1 = 1$.
>
> (b) We need to solve $f'(x) = 0$ using Newton. For the update equation we need $f''(x)$. The update rule becomes $x_{n+1} = x_n - f'(x_n)/f''(x_n)$, plugging in $f'(x) = 4x^3 - 9x^2$ $f''(x) = 12x^2 - 18x$, we get $x_{n+1} = x_n - \frac{x_n(4x_n-9)}{6(2x_n-3)}$. Starting from $x_0 = 0.5$, $x_1 = x_0 - \frac{x_0(4x_0-9)}{6(2x_0-3)}$, we get $0.5 - 0.2916667$ as approximately 0.2083.

4. (a) (3 points) Use Euler's method with $\Delta t = 0.2$ to calculate an approximate solution to $x(0.4)$ for the system with the dynamics defined as $\dot{\mathbf{x}}(t) = t^2 - x(t)^2$, $t > 0$, $x(0) = 1$.

(b) (3 points) What is the approximate solution to $x(0.4)$ when $\Delta t = 0.1$?

(c) (3 points) Given the Butcher tablo below, write the RK equations, is the scheme implicit or explicit?

$$\begin{array}{c|cc} 1/3 & 1/3 & 0 \\ 1 & 1 & 0 \\ \hline & 3/4 & 1/4 \end{array}$$

(d) (3 points) Find the stability function of the scheme represented by the Butcher tablo above.

Solution:

(a) With the standard Euler method $x(t+\Delta t) \approx x_{k+1} = x_k + \Delta t f(x_k)$ the solution to the differential equation $f(x) = \dot{\mathbf{x}} = t^2 - x(t)^2$ is given by $x_{k+1} = x_k + \Delta t f(x_k)$.

For $\Delta t = 0.2$, and $x_0 = x(0) = 1$ we get the sequence

$$\begin{aligned} x_0 &= x(t=0) = 1 \\ x(t=0.2) \approx x_1 &= x_0 + 0.2(0^2 - x_0^2) = 1 - 0.2 = 0.8 \\ x(t=0.4) \approx x_2 &= x_1 + 0.2(0.2^2 - x_1^2) = 0.8 + 0.2.(0.2^2 - 0.8^2) = 0.68 \end{aligned}$$

(b) For $\Delta t = 0.1$, and $x_0 = x(0) = 1$ we get the sequence

$$\begin{aligned} x_0 &= x(t=0) = 1 \\ x(t=0.1) \approx x_1 &= x_0 + 0.1(0-1) = 0.9 \\ x(t=0.2) \approx x_2 &= x_1 + 0.1(0.1^2 - 0.9^2) = 0.82 \\ x(t=0.3) \approx x_3 &= x_2 + 0.1(0.2^2 - 0.82^2) = 0.756 \\ x(t=0.4) \approx x_4 &= x_3 + 0.1(0.3^2 - 0.756^2) = 0.708 \end{aligned}$$

(c) It is implicit as A is not lower diagonal.

$$\begin{aligned} \mathbf{K}_1 &= \mathbf{f}\left(\mathbf{x}_k + \frac{\Delta t}{3} \cdot \mathbf{K}_1, \mathbf{u}(t_k + \frac{\Delta t}{3})\right) \\ \mathbf{K}_2 &= \mathbf{f}\left(\mathbf{x}_k + \Delta t \cdot \mathbf{K}_1, \mathbf{u}(t_k + \Delta t)\right) \\ \mathbf{x}_{k+1} &= \mathbf{x}_k + \frac{3\Delta t}{4}\mathbf{K}_1 + \frac{\Delta t}{4}\mathbf{K}_2 \end{aligned}$$

(d) From the table we have:

$$A = \begin{bmatrix} 1/3 & 0 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}, \quad c = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}.$$

The stability function is $R(\mu) = 1 + \mu b^\top (I - \mu A)^{-1}\mathbf{1}$.

$$I - \mu A = \begin{bmatrix} 1 - \mu/3 & 0 \\ -\mu & 1 \end{bmatrix}$$

Inverting this 2x2 matrix we get

$$(I - \mu A)^{-1} = \begin{bmatrix} \frac{1}{1-\mu/3} & 0 \\ \frac{\mu}{1-\mu/3} & 1 \end{bmatrix}$$

then

$$(I - \mu A)^{-1}\mathbf{1} = \begin{bmatrix} \frac{1}{1-\mu/3} \\ \frac{1+2\mu/3}{1-\mu/3} \end{bmatrix}$$

then

$b^\top (I - \mu A)^{-1}\mathbf{1} = \frac{3}{4}\frac{1}{1-\mu/3} + \frac{1}{4}\frac{1+2\mu/3}{1-\mu/3} = \frac{1+\mu/6}{1-\mu/3}$

and plugging it in the stability function yields:

$R(\mu) = 1 + \mu \frac{1+\mu/6}{1-\mu/3} = \frac{1+2\mu/3+\mu^2/6}{1-\mu/3}$

Appendix: some possibly useful formula

- Lagrange mechanics is built on the equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{Q}, \qquad \mathcal{L}\left(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{z}\right) = \mathcal{T} - \mathcal{V} - \mathbf{z}^\top \mathbf{C}, \qquad \mathbf{C} = 0, \qquad \langle \delta\mathbf{q}, \mathbf{Q} \rangle = \delta W, \, \forall\, \delta\mathbf{q} \tag{1}$$

The kinetic and potential energy of a point mass are given by:

$$\mathcal{T} = \frac{1}{2} m \dot{\mathbf{p}}^\top \dot{\mathbf{p}}, \qquad \mathcal{V} = mg\mathbf{p}_3 \tag{2}$$

respectively, where $\mathbf{p} \in \mathbb{R}^3$ is the position of the mass in a cartesian reference frame having the third coordinate as the vertical axis pointing up. The generalized forces are identical to the external forces applied to a point mass if the position of that point is expressed in cartesian coordinates in the generalized coordinates $\mathbf{q}$.

- In the case $\mathcal{T} = \frac{1}{2} m \dot{\mathbf{q}}^\top W \dot{\mathbf{q}}$ with W constant $\mathcal{V} = \mathcal{V}\left(\mathbf{q}\right)$ and $\mathbf{C} = \mathbf{C}\left(\mathbf{q}\right)$, the Lagrange equations simplify to the dynamics in the semi-explicit index-3 DAE form:

$$\dot{\mathbf{p}} = \mathbf{v} \tag{3a}$$

$$W\dot{\mathbf{v}} + \frac{\partial \mathbf{C}}{\partial \mathbf{q}}^\top \mathbf{z} = \mathbf{Q} - \frac{\partial \mathcal{V}}{\partial \mathbf{q}}^\top \tag{3b}$$

$$0 = \mathbf{C}\left(\mathbf{q}\right) \tag{3c}$$

- The Implicit Function Theorem (IFT) guarantees that a nonlinear set of equations

$$\mathbf{r}\left(\mathbf{y}, \mathbf{z}\right) = 0 \tag{4}$$

"can be solved" in terms of $\mathbf{z}$ for a given $\mathbf{y}$ iff the Jacobian $\frac{\partial \mathbf{r}(\mathbf{y},\mathbf{z})}{\partial \mathbf{z}}$ is full rank at the solution. More specifically, it guarantees that there is a function $\phi\left(\mathbf{y}\right)$ such that

$$\mathbf{r}\left(\mathbf{y}, \phi\left(\mathbf{y}\right)\right) = 0 \tag{5}$$

holds in the neighborhood of the point $\mathbf{y}$ where the Jacobian is evaluated. Furthermore, the IFT specifies that:

$$\frac{\partial \mathbf{z}}{\partial \mathbf{y}} = -\frac{\partial \mathbf{r}}{\partial \mathbf{z}}^{-1} \frac{\partial \mathbf{r}}{\partial \mathbf{y}} \tag{6}$$

- For solving a problem $\mathbf{r}\left(\mathbf{x}\right) = 0$, Newton iterates:

$$\mathbf{x} \leftarrow \mathbf{x} - \alpha \frac{\partial \mathbf{r}}{\partial \mathbf{x}}^{-1} \mathbf{r} \tag{7}$$

until $\mathbf{r}\left(\mathbf{x}\right) \approx 0$ where $\alpha \in [0, 1]$

- Runge-Kutta methods are described by:

$$\begin{array}{c|ccc} c_1 & a_{11} & \dots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \dots & a_{ss} \\ \hline & b_1 & \dots & b_s \end{array}$$

$$\mathbf{K}_j = \mathbf{f}\left(\mathbf{x}_k + \Delta t \sum_{i=1}^{s} a_{ji} \mathbf{K}_i, \, \mathbf{u}\left(t_k + c_j \Delta t\right)\right), \quad j = 1, \dots, s \tag{8a}$$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \sum_{i=1}^{s} b_i \mathbf{K}_i \tag{8b}$$

- For ERK methods, the relationship between the (minimum) number of stages s to the order o is given by:

s	1	2	3	4	6	7	9	11	...
o	1	2	3	4	5	6	7	8	...

Table 1: Stage to order of ERK methods

- Collocation methods use:

$$\dot{\mathbf{x}}(t_k + \Delta t \cdot \tau) \approx \dot{\hat{\mathbf{x}}}(t_k + \Delta t \cdot \tau) = \sum_{i=1}^{s} \mathbf{K}_i \ell_i(\tau), \quad \tau \in [0,\, 1] \tag{9}$$

$$\mathbf{x}(t_k + \Delta t \cdot \tau) \approx \hat{\mathbf{x}}(t_k + \Delta t \cdot \tau) = \mathbf{x}_k + \Delta t \sum_{i=1}^{s} \mathbf{K}_i L_i(\tau) \tag{10}$$

where the Lagrange polynomials are given by:

$$\ell_i(\tau) = \prod_{j=1, j\neq i}^{s} \frac{\tau - \tau_j}{\tau_i - \tau_j}, \quad \text{and} \quad L_i(\tau) = \int_0^{\tau} \ell_i(\xi) \mathrm{d}\xi \tag{11}$$

The Lagrange polynomials satisfy the conditions of

$$\text{Orthogonality:} \quad \int_0^1 \ell_i(\tau)\ell_j(\tau)\, \mathrm{d}\tau = 0 \quad \text{for} \quad i \neq j \tag{12a}$$

$$\text{Punctuality:} \quad \ell_i(\tau_j) = \begin{cases} 1 & \text{if} \quad j = i \\ 0 & \text{if} \quad j \neq i \end{cases} \tag{12b}$$

and enforce the collocation equations (for $j = 1, \ldots, s$):

$$\dot{\hat{\mathbf{x}}}(t_k + \Delta t \cdot \tau_j) = \mathbf{f}\left(\hat{\mathbf{x}}(t_k + \Delta t \cdot \tau_j),\, \mathbf{u}\left(t_k + \Delta t \cdot \tau_j\right)\right), \quad \text{in the explicit ODE case} \tag{13a}$$

$$\mathbf{F}\left(\dot{\hat{\mathbf{x}}}(t_k + \Delta t \cdot \tau_j), \hat{\mathbf{x}}(t_k + \Delta t \cdot \tau_j),\, \mathbf{u}\left(t_k + \Delta t \cdot \tau_j\right)\right) = 0, \quad \text{in the implicit ODE case} \tag{13b}$$

$$\mathbf{F}\left(\dot{\hat{\mathbf{x}}}(t_k + \Delta t \cdot \tau_j), \hat{\mathbf{z}}_j, \hat{\mathbf{x}}(t_k + \Delta t \cdot \tau_j),\, \mathbf{u}\left(t_k + \Delta t \cdot \tau_j\right)\right) = 0, \quad \text{in the fully-implicit DAE case} \tag{13c}$$

- Gauss-Legendre collocation methods select the set of points $\tau_{1,\ldots,s}$ as the zeros of the (shifted) Legrendre polynomial:

$$P_s\left(\tau\right) = \frac{1}{s!}\frac{\mathrm{d}^s}{\mathrm{d}\tau^s}\left[\left(\tau^2 - \tau\right)^s\right] \tag{14}$$

They achieve the order $\|\mathbf{x}_N - \mathbf{x}\left(t_\mathrm{f}\right)\| = \mathcal{O}\left(\Delta t^{2s}\right)$.

- Maximum-likelihood estimation is based on

$$\max_{\boldsymbol{\theta}} \quad \mathbb{P}\left[e_k = y_k - \hat{y}_k \quad \text{for} \quad k = 1, \ldots, N \mid \boldsymbol{\theta}\right] \tag{15}$$

If the noise sequence is uncorrelated, then

$$\mathbb{P}\left[e_k = y_k - \hat{y}_k \quad \text{for} \quad k = 0, \ldots, N \mid \boldsymbol{\theta}\right] = \prod_{k=1}^{N} \mathbb{P}\left[e_k = y_k - \hat{y}_k \mid \boldsymbol{\theta}\,\right] \tag{16}$$

- The solution of a linear least-squares problem

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \frac{1}{2}\left\|A\boldsymbol{\theta} - \mathbf{y}\right\|^2_{\Sigma_e^{-1}} \tag{17}$$

reads as:

$$\hat{\boldsymbol{\theta}} = \left(A^\top \Sigma_e^{-1} A\right)^{-1} A^\top \Sigma_e^{-1} \mathbf{y} \tag{18}$$

and the covariance of the parameter estimation based is given by the formula:

$$\Sigma_{\hat{\boldsymbol{\theta}}} = \left(A^\top \Sigma_e^{-1} A\right)^{-1} \tag{19}$$

- In system identification, given the a plant $G(z)$ and a noise $H(z)$ model description, the one-step-ahead predictor $\hat{y}(k|k-1)$ can be retrieved with

$$H(z)\hat{y}(z) = G(z)u(z) + (H(z)-1)y(z) \tag{20}$$

- The Gauss-Newton approximation in an optimization problem

$$\min_{\mathbf{x}} \quad J(\mathbf{x}) = \frac{1}{2}\left\|\mathbf{R}(\mathbf{x})\right\|^2 \tag{21}$$

uses the approximation:

$$\frac{\partial^2 J}{\partial \mathbf{x}^2} \approx \frac{\partial R}{\partial \mathbf{x}}^\top \frac{\partial R}{\partial \mathbf{x}} \tag{22}$$

- The solution to an LTI system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ is given by:

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)} B\mathbf{u}(\tau)\mathrm{d}\tau \tag{23}$$

and the transformation state-space to transfer function is given by:

$$G(s) = C\left(sI - A\right)^{-1} B + D \tag{24}$$

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $det(A) = ad - bc$
- $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $det(A) = a.det(\begin{bmatrix} e & f \\ h & i \end{bmatrix}) - b.det(\begin{bmatrix} d & f \\ g & i \end{bmatrix}) + c.det(\begin{bmatrix} d & e \\ g & h \end{bmatrix})$
- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $det(A) = ad - bc$, $A^{-1} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- $\alpha = \mathbf{x}^T A\mathbf{x}$, where A is a symmetric matrix and $\mathbf{x}$ is $n \times 1$, A is $n \times n$, and A does not depend on $\mathbf{x}$, then, $\frac{\partial \alpha}{\partial \mathbf{x}} = 2\mathbf{x}^T A$.
- $f(x) = e^{g(x)}$ then $f^{'}(x) = e^{g(x)} g^{'}(x)$